Kinetic Equations

Text of the Exercises

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Recall the Vlasov Equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E(t,x) := -\nabla \left(\frac{1}{|x|} * \rho_t \right)(x), \\ \rho_t(x) := \int_{\mathbb{R}^3} f(t,x,v)\, dv, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1}$$

Exercise 1

Let $f \in C^1_c\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Prove that f satisfies

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} f(y + tv, w) \in L^\infty\left((0,T) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right)\right), \tag{2}$$

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} |\nabla f(y + tv, w)| \in L^\infty\left((0,T) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right) \cap L^2\left(\mathbb{R}^3_v\right)\right). \tag{3}$$

Exercise 2

Let f be a solution of the Vlasov equation (1) such that $\rho \in L^\infty\left((0,T); L^1\left(\mathbb{R}^3\right) \cap L^\infty\left(\mathbb{R}^3\right)\right)$ and with f_0 such that

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} f_0(y + tv, w) \in L^\infty\left((0,T) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right)\right), \tag{4}$$

$$\operatorname*{ess\,sup}_{\substack{y \in B_R(x), \\ w \in B_R(v)}} |\nabla f_0(y + tv, w)| \in L^\infty\left((0,T) \times \mathbb{R}^3_x; L^1\left(\mathbb{R}^3_v\right) \cap L^2\left(\mathbb{R}^3_v\right)\right). \tag{5}$$

Prove that

$$\nabla_v f(t,x,v) \in L^\infty\left((0,T) \times \mathbb{R}^3_x; L^2\left(\mathbb{R}^3_v\right)\right) \tag{6}$$

Hint: Recall that we saw in class that under this hypotheses there exists $\alpha \in (0,1)$ such that $E \in L^\infty\left((0,T); C^{1,\alpha}\left(\mathbb{R}^3\right)\right)$ and use the explicit formula for f given through the characteristics of the problem.

Exercise 3

Let N be a positive integer number and consider ϕ a bounded, continuous and even potential and $\{(x_j, v_j)\}_{j=1}^N \subseteq \mathbb{R}^3 \times \mathbb{R}^3$. Define $\{(x_j(t), v_j(t))\}_{j=1}^N$ as the family of solutions of the Newton equations

$$\begin{cases} \partial_t x_j(t) = v_j(t), \\ \partial_t v_j(t) = -\frac{1}{N}\sum_{k=1}^{N} \nabla\phi\left(x_j(t) - x_k(t)\right), \\ x_j(0) = x_j, \\ v_j(0) = v_j. \end{cases} \tag{7}$$

Consider the measure μ_0 and μ_t given as[1]

$$\mu_0(x,v) := \frac{1}{N}\sum_{j=1}^{N} \delta(x - x_j)\,\delta(v - v_j), \tag{9}$$

$$\mu_t(x,v) := \frac{1}{N}\sum_{j=1}^{N} \delta(x - x_j(t))\,\delta(v - v_j(t)). \tag{10}$$

Prove that μ_t is a weak solution of (1).

[1] Notice that the notation $\delta(x - x_0)$ indicates the delta at the point x_0 and justifies the notation

$$\int_A \delta(x - x_0)\,dx = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A, \end{cases} \qquad \int_{\mathbb{R}^3} \varphi(x)\,\delta(x - x_0)\,dx = \varphi(x_0). \tag{8}$$